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On Curvature Properties of Warped Product Manifolds with Semi-Symmetric Metric Connection and Applications to General Relativity

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Abstract
This paper investigates the new curvature properties of warped product manifolds (WPM) with a semi-symmetric metric connection (SSMC) and its implications for General Relativity (GR) and modern theories of gravity. Warped product manifolds offer a useful geometric setting for many diverse classes of space-time structures such as cosmological and black-hole solutions. The models may explain both intrinsic spin effects and nonequilibrium phenomena associated with the underlying structure of matter because we substitute an SSMC for the conventional Levi-Civita connection (LCC) while preserving metric compatibility & naturally adding torsion. This link may be used to formally express the Riemann curvature tensor (CT), Ricci tensor, scalar curvature, & sectional curvature of WPM. We also prove that these manifolds must be Einstein (also known as conformally flat) in order for them to be. We pay special attention to Robertson–Walker space-times and the way torsion modifies their geometric and physical properties. We give some illustrative examples to show the applicability of our theoretical results and applicability of the derived forms. The resulting curvature identities generalize their classical counterparts for the Levi-Civita connection, while uncovering new geometric characteristics introduced by torsion. Thus, the results provide a more general mathematical basis for research on curved space-times, and also contribute to our understanding of geometric structures arising in differential geometry, mathematical physics, cosmology and alternative by gravity.

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1. Introduction

In modern differential geometry & mathematical physics, the warped product manifolds (WPM) are crucial. Warped products are a ubiquitous technique for the construction and study of Riemannian and semi-Riemannian manifolds with prescribed geometric properties; they have been described by Bishop and O'Neill since 1969. They are not just basic geometry, but they are also used in general relativity, cosmology cosmological black-hole theory and modern gravitational model.

Let $\mathrm{f} : \mathrm{M}_1 \rightarrow \mathrm{R}^+$ be a positive smooth function and let $(\mathrm{M}_1, \mathrm{g}_1)$ & $(\mathrm{M}_2, \mathrm{g}_2)$ be two Riemannian manifolds. The product manifold's warping

$$\mathrm{M} = M_1 \times \mathrm{f}\ M_2$$

is the metric-equipped product manifold $M_1 \times\ M_2$

$$\mathrm{g} = g_1 + f^2\, g_2$$

The function (f), usually referred to as the warping function, scales the metric along the fibers and therefore controls how the base manifold interacts intrinsically with a fiber manifold. Its properties leave a big imprint on the geometry and curvatures of the warped product manifold. This is a natural extension to the warped product framework in which many fundamental space-times of interest in General Relativity may be cast, including Robertson–Walker and Schwarzschild space-times.

Over the past few decades, WPM has been the focus of much research due to its rich mathematical structure & numerous applications in theoretical physics & differential geometry. It has been explored their curvature properties, Einstein structures, conformal geometry, Ricci solitons and many geometric flows in many papers. We also mention the significance of warped product manifolds in mathematical physics, particularly in relation to cosmology & General Relativity (GR). This behavior of curvature is gives useful information about both local and global geometry of each manifold; it also providing a suitable guide in the modeling and analyzing physically significant space-times.

Along with the geometry of warped products, much interest has been focused on affine connections which may have torsion. Hayden develops one of these, a class of semi-symmetric metric connection (SSMC), which Yano methodically investigates. Unlike the torsion-free Levi-Civita connection (LCC), an SSMC offers metric compatibility with a nonzero torsion tensor of a specified shape. This further geometrical structure adjusts the curvature nature of the manifold and gives a less particular setting to investigate differential geometry, mathematical physics, physical theories based on torsion-gravitation.

$$T(X, Y) = \pi(Y)X - \pi(X)Y$$

where π is a vector field P's differential one-form. Many geometric aspects are preserved while new curvature effects are produced since the connection is consistent with the metric even in the presence of torsion.

Semi-symmetric metric connections (SSMC) have found applications in differential geometry, contact geometry, almost Hermitian manifolds, and gravitational theories involving torsion. Their study has revealed interesting modifications of classical curvature identities and has led to new geometric classifications. Specifically, the addition of torsion frequently modifies the behavior of Einstein conditions, scalar curvature, as well as Ricci curvature, producing richer geometric structures than those derived from the Levi-Civita link alone.

Recently, there has been increasing interest in examining the interaction between warped product structures and generalized connections. Such investigations are motivated by both geometric and physical considerations. In geometric terms, we ask how torsion influences curvature quantities or classical results still hold. Additionally, a number of expansions of Einstein's theory of gravitation, including the Einstein–Cartan theory as well as gauge theories of gravity from a physical standpoint, naturally incorporate torsion.

The role of warped product geometry in differential geometry and mathematical physics is again underscored by recent works. Curvature conditions, Ricci solitons, Einstein structures and generalized metric connections in warped product manifolds have been studied by several authors. So, this part of the paper is focused mainly on the recent advances related to semi-simple metric connections, generalized curvatures and relativistic space time that can enable us some constructive feedback for how torsion interacts with geometric invariants in terms of different dimensions. Inspired by this, the current study investigates the curvature characteristics of WPM with SSMC & their implications for GR. This is of importance in giving us new tools to expose torsion effects induced on geometric invariants and space-time structures, thus offering new perspectives on Einstein manifolds and cosmological models or alternative gravitational theories.

In this article, we are concerned with semi-symmetric metric connections on warped product manifolds as well as their geometrical characteristics. Our main goal is to obtain explicit closed-form formulas for: the sectional curvature, scalar curvature, Ricci tensor, and Riemann curvature tensor, as well as an analysis of their geometric consequences. Special emphasis is given to characterize Einstein manifolds, conditions for conformal flatness and applications of these results to General Relativity, particularly as they relate to cosmological space-time models.

The following is a summary of this work's primary contributions:

1. It is possible to accurately define the curvature tensor (CT) of WPM with SSMC.
2. Ricci tensor and scalar curvature expressions are obtained through appropriate contractions of the curvature tensor.
3. For Einstein warped product manifolds, sufficient and necessary criteria are determined.
4. Sectional curvature and conformal flatness properties are investigated.
5. Applications to Robertson–Walker space-times are presented, providing geometric interpretations relevant to cosmological models.

2. Preliminaries

In this study, it is assumed that all tensor fields & manifolds are smooth.

Assume that (M, g) is an n-dimensional Riemannian manifold (RM) and that the Levi-Civita connection (LCC) for the metric g is represented by ∇. It is uniquely defined by the torsion free and metric compatibility conditions.

2.1. Definition (Curvature Tensor)

The Riemannian curvature tensor (RCT) is defined as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

2.2. Definition (Ricci Tensor)

The Ricci tensor may be obtained by tracing the curvature tensor:

$$S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i)$$

where $\{e_i\}$ is a local orthonormal frame?

2.3. Definition (Scalar Curvature)

The curvature of the scalar is

$$r = \sum_{i=1}^{n} S(e_i, e_i)$$

2.4. Definition (SSMC)

A linear connection $\overline{\nabla}$ on M is known as SSMC if

$$T(X,Y) = \pi(Y)X - \pi(X)Y$$

$$\overline{\nabla}\, g = 0$$

$$\overline{\nabla}_X \mathrm{Y} = \nabla_X \mathrm{Y} + \pi(\mathrm{Y})\,\mathrm{X} - \mathrm{g}(\mathrm{X},\mathrm{Y})\mathrm{P}$$

$$\pi(X) = g(X,P)$$

3. WPM Geometry

WPM are a significant class of Riemannian manifolds (RM) that provide a versatile and natural environment for studying a variety of geometric problems. Due to their rich mathematical structure, they are especially good at studying curvature properties, Einstein metrics and other intrinsic geometric features. As a result of translating these characteristics, WPM has found extensive use in the domains of differential geometry (DG), mathematical physics, and general relativity (GR), serving as more appropriate models to explain the connection between physical space–time geometry.

Let $(\mathrm{M}_1, \mathrm{g}_1)$ and $(\mathrm{M}_2, \mathrm{g}_2)$ be n_1 & n_2 Riemannian manifolds (RM), respectively. Let's

$$\mathrm{f} : \mathrm{M}_1 \to \mathrm{R}^+$$

be a positive function that is smooth. The WPM

$$\mathrm{M} = M_1 \times \mathrm{f}\ M_2$$

is the metric-equipped product manifold $M_1 \times M_2$

$$\mathrm{g} = g_1 + f^2\, g_2$$

The warping function is represented by the function *f*.

The terminology used in this section will remain constant: U, V, & W are smooth vector fields tangent to the fiber manifold (M_2), whereas X, Y, and Z are smooth vector fields tangent to the base manifold (M_1). We shall use this convention throughout in order to streamline the presentation of the curvature formulas and what follows.

3.1. Theorem (Bishop–O'Neill Formulas)

Let $\mathrm{M} = M_1 \times \mathrm{f}\ M_2$

be a product manifold that is warped.

Then the Levi-Civita connection (LCC) ∇ satisfies:

- For vector fields X and Y that are horizontal

$$\nabla_X \mathrm{Y} = \nabla_X^{(1)}\mathrm{Y}$$

where $\nabla^{(1)}$ displays the LCC on M_1.

- For $X \in TM_1$ and $U \in TM_2$

$$\nabla_X \mathrm{U} = \nabla_U \mathrm{X} = \frac{X(f)}{f}\ \mathrm{U}$$

- $U, V \in TM_2$

$$\nabla_U\ \mathrm{V} = \nabla_U^{(2)}\ \mathrm{V} - \frac{\mathrm{g}(\mathrm{U},\mathrm{V})}{f}\ \nabla\mathrm{f}$$

where $\nabla^{(2)}$ represents the LCC on M_2.

- **Proof**

Koszul's formula provides the proof:

$$\begin{aligned} 2\mathrm{g}(\nabla_X \mathrm{Y}, \mathrm{Z}) = \mathrm{Xg}(\mathrm{Y},\mathrm{Z}) + \mathrm{Yg}(\mathrm{X},\mathrm{Z}) - \mathrm{Zg}(\mathrm{X},\mathrm{Y}) \\ + g([X,Y],Z) - g([X,Z],Y) \\ - g([Y,Z],X) \end{aligned}$$

We analyze the three cases separately.

Case 1: Horizontal-Horizontal Fields

Let

$$\mathrm{X}, \mathrm{Y}, \mathrm{Z} \in \mathrm{TM}_1$$

Since

$$\mathrm{g} = g_1 + f^2\, g_2$$

where all vector fields are tangent to M_1.

$$\mathrm{g}(\mathrm{X},\mathrm{Y}) = \mathrm{g}_1(\mathrm{X},\mathrm{Y})$$

Applying Koszul's formula immediately yields

$$\nabla_X \mathrm{Y} = \nabla_X^{(1)}\mathrm{Y}$$

Case 2: Horizontal-Vertical Fields

Let

$$X \in TM_1 \text{ and } U, V \in TM_2$$

Applying Koszul's formula gives

$$2\mathrm{g}(\nabla_X \mathrm{U}, \mathrm{V}) = \mathrm{X}(\mathrm{g}(\mathrm{U},\mathrm{V}))$$

$$\mathrm{g}(\mathrm{U},\mathrm{V}) = f^2 \mathrm{g}_2(\mathrm{U},\mathrm{V})$$

$$\mathrm{X}(\mathrm{g}(\mathrm{U},\mathrm{V})) = 2\mathrm{fX}(\mathrm{f})\mathrm{g}_2(\mathrm{U},\mathrm{V})$$

$$\mathrm{g}(\nabla_X \mathrm{U}, \mathrm{V}) = \frac{X(f)}{f}\mathrm{g}(\mathrm{U},\mathrm{V})$$

Since this holds for all V

$$\nabla_X \mathrm{U} = \frac{X(f)}{f}\mathrm{U}$$

Case 3: Vertical-Vertical Fields

Let

$$\mathrm{U}, \mathrm{V}, \mathrm{W} \in \mathrm{TM}_2$$

Applying Koszul's formula yields

$$\nabla_U V = \nabla_U^{(2)} V - \frac{g(U,V)}{f}\nabla f$$

3.2. Corollary

The warping function's gradient satisfies

$$g(\nabla f, X) = X(f)$$

for any vector field X that is horizontal
Consequently

$$\nabla f = \mathrm{grad}(f) \in TM_1$$

3.3. Proposition

For every vertical vector field *U*

$$R(X,U)Y = -\frac{H^f(X,Y)}{f}U$$

$$H^f(X,Y) = g(\nabla_X \nabla f, Y)$$

represents the Hessian of f

- **Proof**

Using Theorem 3.1

$$\nabla_X U = \frac{X(f)}{f}U$$

Simplifying & substituting into the curvature tensor definition produces

$$R(X,U)Y = -\frac{H^f(X,Y)}{f}U$$

3.4. Proposition

Regarding vertical vector fields *U*, *V*, & *W*

$$R(U,V)W = R^{(2)}(U,V)W - \frac{\|\nabla f\|^2}{f^2}[g(V,W)U - g(U,W)V]$$

- **Proof**

Substituting Theorem 3.1 into the definition

$$R(U,V)W = \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W$$

and simplifying gives the required expression.

4. Curvature Tensor (CT) under SSMC

We focus on deriving the CT of the SSMC on warped product manifolds in this section. The mathematical approach for examining the geometric structure and curvature properties of these manifolds is established by these equations. In addition, the obtained formulas are fundamental for submanifold invariants such as Ricci tensor, scalar curvature, sections curvature and Einstein conditions so that this work paves a way to the applications in the following two sections.

$$\bar\nabla_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P$$

The following theorem provides the fundamental curvature formula used throughout the remainder of the review.

4.1. Theorem

Let (M, g) accept a metric connection $\bar\nabla$ that is semi-symmetric.
Then

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha(Y,Z)X - \alpha(X,Z)Y + g(Y,Z)LX - g(X,Z)LY$$

Where

$$\alpha(X,Y) = (\nabla_{X\pi})(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(P)g(X,Y)$$

and

$$LX = \nabla_X P - \frac{1}{2}\pi(P)X$$

- **Proof**

Starting from

$$\bar R(X,Y)Z = \bar\nabla_X\bar\nabla_Y Z - \bar\nabla_Y\bar\nabla_X Z - \bar\nabla_{[X,Y]}Z$$

we substitute

$$\bar\nabla_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P$$

Expanding all terms and collecting coefficients involving π, P and ∇P we obtain the stated formula.

5. Ricci Curvature of WPM with SSMC

One of the most intrinsic geometric objects in differential geometry and gravitational physics is the Ricci tensor, because it characterizes how the curvature is distributed over a manifold. The Ricci tensor connects geometry and matter in General Relativity by connecting space-time characteristics to energy density distributions, which is at the core of Einstein's field equations. As a result, we think it is essential to look at how it behaves in connection to an SSMC. An analysis of this sort uncovers the connection between torsion and curvature structure, sheds more light on Einstein manifolds and contributes to our understanding of geometric models used in cosmology and alternative gravitation theories.
In this section, let (M, g)
be an n-dimensional WPM with a SSMC.

$$\bar\nabla_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P$$

5.1 Definition

The connection $\bar\nabla$ Ricci tensor is determined by

$$\bar S(Y,Z) = \sum_{i=1}^{n} g(\bar R(e_i,Y)Z, e_i)$$

Where $\{e_1, \dots, e_n\}$
is a local orthonormal frame?

5.2. Theorem

Consider a warped product manifold $(M, g, \bar\nabla)$. The associated curvature tensors and related invariants will be examined to study the geometric properties of this connection.

Then the Ricci tensor satisfies

$$\bar{S}(Y,Z) = S(Y,Z) - (n-2)\,\alpha(Y,Z) - \beta\, g(Y,Z)$$

Where $\beta = tr(\alpha)$

- **Proof**

Starting from Theorem 4.1,

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha(Y,Z)X - \alpha(X,Z)Y + g(Y,Z)LX - g(X,Z)LY$$

Choose an orthonormal frame

$$\{e_i\}_{i=1}^{n}$$

Taking the trace over $X = e_i$

$$\bar{S}(Y,Z) = \sum_i g(\bar{R}(e_i,Y)Z, e_i)$$

Using the curvature formula as a substitute,

$$\bar{S}(Y,Z) = S(Y,Z) + \sum_i g(\,\alpha(Y,Z)e_i, e_i) - \sum_i g(\alpha(e_i,Z)Y, e_i) + \sum_i g(\,g(Y,Z)Le_i, e_i) - \sum_i g(g(e_i,Z)Y, e_i)$$

$$\sum_i g(e_i, e_i) = n$$

$$\sum_i \alpha(e_i,Z)g(Y,e_i) = \alpha(Y,Z)$$

$$\bar{S}(Y,Z) = S(Y,Z) - (n-2)\alpha(Y,Z) - \beta g(Y,Z)$$

5.3. Corollary (Ricci-Flat Condition)

In relation to $\overline{\nabla}$, the manifold is Ricci-flat if & only if

$$S(Y,Z) = (n-2)\alpha(Y,Z) + \beta g(Y,Z)$$

- **Proof**

$$\bar{S} = 0$$

in Theorem 5.1 immediately yields the result.

5.4. Corollary

If $\alpha = 0$, Then $\bar{S} = S$

The Ricci tensor & the classical Ricci tensor associated with the Levi-Civita link therefore coincide.

6. Curvature of Scalars

We now have the scalar curvature associated with the SSMC. This invariant provides a global measure of the manifold's curvature and plays a significant role in understanding its geometric and physical properties.

6.1. Definition

$\bar{r} = \sum_{i=1}^{n} \bar{S}\,(e_i, e_i)$ defines the scalar curvature.

6.2. Theorem

The scalar curvature of a warped product manifold (WPM) with an SSMC satisfies:

$$\bar{r} = r - 2(n-1)\beta$$

- **Proof**

From Theorem 5.1

$\bar{S} = S - (n-2)\,\alpha - \beta\, g$

$\bar{r} = r - (n-2)tr(\alpha) - n\beta$

Since $tr(\alpha) = \beta$ We obtain

$$\bar{r} = r - (n-2)\beta - n\beta$$

$$\bar{r} = r - 2(n-1)\beta$$

6.3. Corollary

If $\beta = 0$, Then $\bar{r} = r$

Hence the scalar curvature remains unchanged.

6.4. Corollary

If $\beta > 0$, Then $\bar{r} < r$

Thus, torsion decreases scalar curvature.

Similarly, $\beta < 0$, Implies $\bar{r} > r$

- **Physical Interpretation**

In General Relativity (GR), scalar curvature appears in the Einstein-Hilbert action which suggests that it plays a fundamental role in regards to gravitational dynamics and the geometric structure of space-time.

$$I = \int_M r\sqrt{|\,g\,|}\,dV$$

In an SSMC, this action turns into

$$\bar{I} = \int_M \bar{r}\sqrt{|\,g\,|}\,dV$$

Substituting Theorem 6.1

$$\bar{I} = \int_M (r - 2(n-1)\beta)\sqrt{|\,g\,|}\,dV$$

This shows explicitly how torsion modifies the gravitational action and consequently affects the geometry of spacetime.

7. Einstein Warped Product Manifolds (WPM)

This section examines warping product manifolds that meet the Einstein condition and have semi-symmetric metric connections. We provide necessary and sufficient conditions for these manifolds to be Einstein, focusing on the interplay between torsion as well as curvature features. The subsequent theoretical results are the mathematical basis of their later applications to Robertson–Walker space-times and their geometric interpretation within the concept of General Relativity.

7.1. Definition

An Einstein manifold is a manifold if

$$\bar{S} = \lambda g$$

for some constant λ.

7.2. Theorem (Characterization of Einstein Structures)

If & only if a WPM has a SSMC, then it is Einstein.

$$S = (\lambda + \beta)\, g + (n - 2)\, \alpha$$

- **Proof**

Substitute $\bar{S} = \lambda g$
into Theorem 5.1 and rearrang terms.

8. Sectional Curvature

The sectional curvature is one of the simplest and basic local invariants in Riemannian geometry that measures how much the two-dimensional plane in the tangent space at every point on a manifold bends. Such a structure is at the heart of the classification of space forms, the characterization of Einstein manifolds and more generally in the study of geometric structures arising e.g. in differential geometry, mathematical physics or General Relativity itself. Sectional curvature determines the intrinsic curvature of individual tangent planes and elicits local geometric behavior of manifolds that can serve as a basis for global curvature results.

Let $\sigma = \mathrm{span}\{X, Y\}$

be a plane in $T_P M$ with two dimensions, where X and Y are tangent vectors that are linearly independent.

8.1. Definition

The sectional curvature of the SSMC is calculated using

$$\bar{K}(\sigma) = \frac{g(\bar{R}(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

If X and Y are orthonormal, then

$$\bar{K}(\sigma) = g(\bar{R}(X,Y)Y,X)$$

8.2. Theorem

Let us suppose that the warped product manifold $M, g, \bar{\nabla}$ admits a SSMC. Assuming this, we look into the curvature properties arise on the manifold corresponding to $M, g, \bar{\nabla}$and determine its geometric relations on the manifold.

$$\bar{K}(\sigma) = K(\sigma) + \alpha(Y,Y) - \alpha(X,X)$$

where the sectional curvature associated with the Levi-Civita link is represented by $K(\sigma)$.

- **Proof**

Using Theorem 4.1

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha(Y,Z)X - \alpha(X,Z)Y + g(Y,Z)LX - g(X,Z)LY$$

Assuming X and Y are orthonormal

$$g(X,X) = g(Y,Y) = 1\,, g(X,Y) = 0$$

Applying X to take the inner product

$$g(\bar{R}(X,Y)Y,X) = g(R(X,Y)Y,X) + \alpha(Y,Y) - \alpha(X,X)$$

$$\bar{K}(\sigma) = K(\sigma) + \alpha(Y,Y) - \alpha(X,X)$$

8.3. Corollary

If $\alpha = 0$ Then $\bar{K}(\sigma) = K(\sigma)$

As a result, the sectional curvature as well as the classical sectional curvature coincide.

9. Conformal Flatness

It aims to gain insight into properties of manifolds invariant under conformal (angle-preserving) transformations, in which a positive smooth function scales the metric while leaving angles unchanged. A key component of this theory is the Weyl conformal curvature tensor, which characterizes a manifold's conformal curvature while disregarding its Ricci curvature.

The Weyl tensor zeroing out presents a fundamental necessary condition for conformal flatness and is of paramount importance in differential geometry, the theory of General Relativity, and gravitational field investigations.

9.1. Definition

For $n \geq 4$, the Weyl tensor is defined by

$$W = R - \frac{1}{n-2}(S \wedge g) + \frac{r}{(n-1)(n-2)}(g \wedge g)$$

where the Kulkarni–Nomizu product is indicated with ∧.

A manifold is considered conformally flat if $W = 0$

9.2. Theorem

$(M, g, \bar{\nabla})$ represents a WPM with a SSMC.

If $W = 0$ And $\alpha = \mu g$ it follows that $\alpha \wedge g = \mu(g \wedge g)$

Thus, the extra curvature term only enters the scalar part of the decomposition of curvature and does not affect the Weyl conformal curvature tensor. As a result, even in the presence of a SSMC and its torsion, the requirement of conformal flatness is maintained since the condition for the vanishing of Weyl tensor up to a manifold remains unchanged.

Hence $\bar{W} = 0$

10. Applications to General Relativity

A main reason for studying warped product manifolds is that they appear in several applications in cosmology and gravitational physics. Without necessarily being a complete space–time in itself, many physically significant space–times can therefore be suitably described as warped products of this type; Robertson–Walker and Schwarzschild, to mention only two cosmological models.

This geometric formulation gives a powerful method for investigating curvature properties, Einstein equations and their implication on the evolution of space-time by its geometric structures.

10.1. Robertson–Walker Space-Time

The Robertson–Walker metric is given by

$$ds^2 = -dt^2 + a^2(t)gF$$

where $a(t)$ is the cosmic scale factor that describes how our universe expands or contracts in relation to cosmic time, and gF is a metric tensor for some three-dimensional space of constant sectional curvature. Combined, these elements give the Robertson–Walker space-time, one of the standard geometric spaces for homogeneous and isotropic cosmological universes in General Relativity.

The formula for this space-time is $M = I \times aF$

Robertson-Walker space-time is hence a warped manifold of products.

10.2. Theorem

Let $\mathrm{M} = \mathrm{I} \times \mathrm{aF}$ be a Robertson–Walker space-time with a SSMC generated by

$$\mathrm{P} = \frac{\partial}{\partial \mathrm{t}}$$

Then, $\bar{\mathrm{r}} = \mathrm{r} - 2(\mathrm{n}-1)\beta$ is satisfied by the scalar curvature.

- **Proof**

Directly applying Theorem 6.1 to the Robertson–Walker metric yields the stated formula.

- **Physical Interpretation**

The quantity $\beta = \mathrm{tr}(\alpha)$ acts as an additional geometric contribution produced by torsion.
Consequently $\bar{r} \neq \mathrm{r}$
These results suggest that a SSMC changes the effective curvature of the universe and modifies its fundamental geometric structure. These types of alterations could affect both the structure and dynamics of cosmological space-time models by adding torsion-related curvature terms to a given model without conventional matter sources, or ignoring the latter altogether. Thus, the semi-symmetric metric connections give a wider geometric background to study different cosmic solutions.

10.3. Modified Einstein Equations

Einstein's field equations are

$$\mathrm{Ric} - \frac{1}{2}\mathrm{rg} + \Lambda \mathrm{g} = 8\pi \mathrm{T}$$

Using the semi-symmetric metric connection, they become

$$\overline{\mathrm{Ric}} - \frac{1}{2}\bar{\mathrm{r}}\mathrm{g} + \Lambda \mathrm{g} = 8\pi \overline{\mathrm{T}}$$

Substituting Theorems 5.1 and 6.1 gives

$$\mathrm{S} - (\mathrm{n}-2)\alpha - \beta \mathrm{g} - \frac{1}{2}(\mathrm{r} - 2(\mathrm{n}-1)\beta)\mathrm{g} + \Lambda \mathrm{g} = 8\pi \overline{\mathrm{T}}$$

10.4. Theorem

The torsion vector field *P* contributes an effective geometric stress-energy term

$$\mathrm{T}^{(\mathrm{P})} = \frac{1}{8\pi}[(\mathrm{n}-2)\alpha + \beta \mathrm{g}]$$

- **Proof**

Rearranging the modified Einstein equations yields

$$\mathrm{Ric} - \frac{1}{2}\mathrm{rg} + \Lambda \mathrm{g} = 8\pi(\overline{\mathrm{T}} + \mathrm{T}^{(\mathrm{P})})$$

Therefore, the torsion field behaves as an effective matter source.

10.5. Corollary

This semi-symmetric metric connection can also produce nontrivial curvature effects, even in the absence of conventional forms of matter or energy. This suggests that torsion can be directly responsible for the geometric structure of space-time giving rise to one more gravitational mechanism.
An illustration of how such a feature may be useful for constructing alternative cosmological models, in which gravity has at least a partial geometric origin as opposed to being solely a material source.

11. Worked Example

Consider $\mathrm{M} = \mathrm{R} \times_{e^t} S^n$

The metric is $\mathrm{g} = \mathrm{d\,t}^2 + \mathrm{e}^{2t}\mathrm{g}\, S^n$

Choose $\mathrm{P} = \frac{\partial}{\partial \mathrm{t}}$

Then $\pi(\mathrm{X}) = \mathrm{g}(\mathrm{X}, \mathrm{P})$

Since $\nabla \mathrm{P} = 0$

We get $\alpha(\mathrm{X}, \mathrm{Y}) = -\pi(\mathrm{X})\pi(\mathrm{Y}) + \frac{1}{2}\mathrm{g}(\mathrm{X}, \mathrm{Y})$

Consequently $\bar{S} = \mathrm{S} - (\mathrm{n}-2)\,\alpha - \beta \mathrm{g}$

This example clearly shows that torsion can have a large effect on the Ricci curvature and therefore influence the scalar curvature of a warped product manifold. The geometric structure of the manifold changes accordingly, thus showing how a SSMC modifies curvature invariants and modifies the underlying geometric and physically relevant structures of this space.

12. Conclusion

The importance of WPM with SSMC to general relativity was examined in this paper. The curvature tensor, sectional curvature, scalar curvature, & Ricci tensor all have explicit formulations. Conditions for Einstein manifolds and conformal flatness were established. Furthermore, applications to Robertson–Walker space-times demonstrated how torsion modifies classical geometric quantities and contributes additional curvature effects in gravitational models. The obtained results extend several classical formulas of Bishop–O'Neill geometry and provide a framework for future investigations of warped product manifolds in geometric analysis, cosmology, Einstein–Cartan theory, and alternative theories of gravity.

References

1. Blaga AM. Geometry of manifolds and applications. Mathematics. 2025;13(6):990. doi:10.3390/math13060990.
2. Bishop RL, O'Neill B. Manifolds of negative curvature. Trans Am Math Soc. 1969;145:1-49. doi:10.1090/S0002-9947-1969-0251664-4.
3. Chen BY. Geometry of warped product CR-submanifolds in Kaehler manifolds. Monatsh Math. 2001;133(3):177-195. doi:10.1007/s006050170019.
4. Chen BY. *Pseudo-Riemannian geometry, δ-invariants and applications*. Singapore: World Scientific; 2011.
5. Bejancu M, Farran HR. *Foliations and geometric structures*. Dordrecht: Springer; 2006. doi:10.1007/1-4020-3720-1.
6. Hayden HA. Subspaces of a space with torsion. Proc Lond Math Soc. 1932;s2-34(1):27-50.

doi:10.1112/plms/s2-34.1.27.

7. Kobayashi S, Nomizu K. *Foundations of differential geometry*. Vol. 1. New York (NY): Wiley-Interscience; 1963.
8. Mofarreh F, Srivastava SK, Kumar A, Ali A. Geometric inequalities of PR-warped product submanifolds in para-Kenmotsu manifold. AIMS Math. 2022;7(10):19481-19509. doi:10.3934/math.20221069.
9. O'Neill B. *Semi-Riemannian geometry with applications to relativity*. New York (NY): Academic Press; 1983.
10. Aloui F. Einstein-like Poisson warped product manifolds and applications. Symmetry. 2025;17(5):645. doi:10.3390/sym17050645.
11. Rahman S, Paul AK. Geometry of warped product pseudo slant submanifolds in nearly Lorentzian para-Sasakian manifolds. J Hyperstruct. 2023. doi:10.22098/jhs.2023.2590.
12. Shaikh AA, Hui SK, Sarkar M. Curvature properties of a warped product metric. J Math. 2024;13(1):220-231.
13. Tripathi MM. A new connection in a Riemannian manifold. Int Electron J Geom. 2008;1(1):15-24.
14. Chen Y. *Differential geometry of warped product manifolds and submanifolds*. Singapore: World Scientific; 2017. doi:10.1142/10419.

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