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Consequence of Fixed Point in Controlled Metric Space with Application

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Abstract
In this paper, we deduce certain generalized fixed point results as a consequence of our main results and obtain some common fixed point theorems for generalized contractions using specific control functions in controlled metric space. Additionally, Numerous well-known findings from the literature will be modified and generalized by our findings. To demonstrate the validity of the stated results, we also offer an example. We study the solution of integral equations as an application of our fundamental finding.

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Introduction

In 1906, M. Frechet provided a basic explanation of the manifest evolution of a metric space. In recent years, numerous academics have expanded and generalized this concept. Examples include complex valued metric space, cone metric space, 7-metric space, Θ-metric space, orthogonal metric space, extended b-metric space, b-metric space, and controlled metric space.

Czerwik [1] gave the notion of a *b*-metric space as follows:

Definition 1. (see.[1]) Let $R \neq \emptyset$ and $s \geq 1$ and $\tau: R \times R \to [0,\infty)$.If:

(τ1) $\tau(\mu,\varsigma) = 0 \Leftrightarrow \mu=\varsigma$;

(τ2) $\tau(\mu,\varsigma) = \tau(\varsigma,\mu)$ for all $\mu,\varsigma \in R$;

(τ3) $\tau(\mu,\omega) \leq s[\tau(\mu,\varsigma)+\tau(\varsigma,\omega)]$ for all $\mu, \varsigma, \omega \in R$.

Then (R,τ) is said to be a b-metric space.

Kamran *et al*.[2]defined the notion of a extended *b*-metric space in 2017:

Definition 2. Let $R \neq \emptyset$ and $\sigma: R \times R \to [1,\infty)$ and $\tau: R \times R \to [0,\infty)$.If

$\tau(\mu,\varsigma)= 0 \Leftrightarrow \mu=\varsigma$;

$\tau(\mu,\varsigma)=\tau(\varsigma,\mu)$;

$\tau(\mu,\varsigma) \leq \sigma(\mu,\varsigma)[\tau(\mu,\omega)+\tau(\varsigma, \omega)]$.

Then (R, τ) is said to be an extended b-metric space.
In 2018, a new type of extended *b*-metric space was given by Mlaiki *et al.* [3]:

Definition 3([3]).Let R/=∅ and σ:R×R→[1,∞) and τ:R×R→[0,∞).If:

$\tau(\mu,\varsigma) = 0 \Leftrightarrow \mu=\varsigma$;

$\tau(\mu,\varsigma) = \tau(\varsigma,\mu)$;

$\tau(\mu,\varsigma) \leq \sigma(\mu,\omega)\tau(\mu,\omega)+\sigma(\varsigma,\omega)\,\tau(\varsigma,\omega)$.

Then (R,τ,σ) is said to be a controlled metric space.

Example1 ([3]).*Let* R={1,2,⋯}. *Take* τ:R×R→[0,∞)*as*

$$\tau(\mu,\varsigma) = \begin{Bmatrix} 0, if\ \mu = \varsigma \\ \frac{1}{\mu}, if\ \mu\ is\ even\ and\ \varsigma\ is\ odd \\ \frac{1}{\varsigma}, if\ \mu\ is\ odd\ and\ \varsigma\ is\ even \\ 1, otherwise \end{Bmatrix}.$$

consider σ:R×R→[1,∞) as

$$\sigma(\mu,\varsigma) = \begin{Bmatrix} \mu, if\ \mu\ is\ even\ and \varsigma\ is\ odd \\ \varsigma, if\ \mu\ is\ odd\ and\ \varsigma\ is\ even \\ 1, otherwise \end{Bmatrix}.$$

Then, τ is a controlled metric and (R,τ,σ) is a controlled metric space.

Theorem 1([3]). Let (R, σ, τ)be a complete controlled metric space. Let V:R→R be such tha*t:*
$\tau(V\mu, V\varsigma) \leq \lambda(\tau(\mu,\varsigma))$

$$\sup \lim_{i\to\infty} \frac{\sigma(\mu_{i+1}, \mu_{i+2})\,\sigma(\mu_{i+1}, \mu_m)}{\sigma(\mu_i\, \mu_{i+1})} < \frac{1}{\lambda}$$

Furthermore, suppose that, ∀μ ∈R, we have $\lim_{n\to\infty} \sigma(\mu_n,\mu)$ and $\lim_{n\to\infty}\sigma(\mu,\mu_n)$ which exist and are finite. Then, ∃ μ* ∈R such that Vμ* =μ*which is unique.
Lateef [4] established Kannan [5] type fixed point theorems in the setting of a controlled metric space.

Theorem 2 ([4]).Let (R, σ, τ) be a complete controlled metric space. Let V:R→R be such t*hat:*

$\tau(V\mu, V\varsigma) \leq \lambda(\tau(\mu,V\mu) + \tau(\varsigma,V\varsigma))$

$\forall \mu,\varsigma \in$ R, where $\lambda \in (0,\frac{1}{2})$. For $\mu_0 \in$ R, take $\mu_n=V^n\mu_0$. Assume that:

$$sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\mu_{i+1},\mu_{i+2})\sigma(\mu_{i+1},\mu_m)}{\sigma(\mu_i,\mu_{i+1})} < \frac{1}{\lambda}$$

Furthermore, suppose that, ∀ μ ∈ R, we have: $\lim_{n\to\infty}\sigma(\mu_n, \mu)$ and $\lim_{n\to\infty}\sigma(\mu,\mu_n)$ exist and are finite. Then, ∃μ* ∈R such that Vμ* =μ*which is unique.
Ahmad [6] established a Reich type fixed-point theorem in the setting of controlled metric space as follows.

Theorem 3. Let (R,σ,τ) be a complete controlled metric space and V:R→R.If there exists α, β, γ ∈ (0,1) such that λ= α + β + γ < 1and:

$\tau(V\mu, V\varsigma) \leq \alpha\, \tau(\mu,\varsigma) + \beta\, \tau(\mu,V\mu) + \gamma\, \tau(\varsigma,V\varsigma)$

$\forall\mu, \varsigma \in$ R. *For* $\mu_0 \in$R, *take* $\mu_n=V^n\mu_0$. *Suppose that:*

$$sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\mu_{i+1},\mu_{i+2})\sigma(\mu_{i+1},\mu_m)}{\sigma(\mu_i,\mu_{i+1})} < \frac{1}{\lambda}$$

Furthermore, suppose that, for every $\mu \in R$, we have $\lim_{n\to\infty} \sigma(\mu_n,\mu)$ and $\lim_{n\to\infty}\sigma(\mu,\mu_n)$ Which exist and are finite. Then, $\exists \mu^* \in R$ such that $V\mu^*=\mu^*$ which is unique.
Later on, Abuloha *et al*. [7], Alamgir *et al*. [8], Abdeljawad *et al*. [9], Lateef [10], Hussain [11], Mlaikietal. [12], Shatanawi *et al*. [13], Sezenetal. [14] and Tasneem *et al*. [15] studied controlled metric spaces and established different fixed-point results for self and multi valued mappings. For more details, in this direction, we refer the readers to [1–17].
In this paper, we obtain some common fixed-point results for generalized contractions involving some certain control functions in the setting of controlled metric spaces. We also proved some common fixed-point theorems in controlled metric spaces endowed with a graph. We also provided an example to show the legitimacy of the established results. As an application of our main result, we investigate the solution of integral equations.

Main Results
We state our main result as follows.

Theorem 4. Let (R,σ,τ) be a complete controlled metrics space and $V,U:R\to R$. If there exists α, β: $R\to [0,1)$ such that:

$$\alpha(V\mu)\leq\alpha(\mu),\ \beta(V\mu)\leq\beta(\mu) \text{ and } \gamma(V\mu)\leq \gamma(\mu)$$

$$\alpha(U\mu)\leq\alpha(\mu),\ \beta(U\mu)\leq\beta(\mu) \text{ and } \gamma (U\mu) \leq\gamma (\mu)$$

$$(\alpha+\beta+\gamma)(\mu)<1;$$

$$\tau(V\mu, U\varsigma)\leq \alpha(\mu)\tau(\mu,\varsigma)+\beta \frac{\tau(\mu,V\mu)\tau(\varsigma,U\varsigma)}{1+\tau(\mu,\varsigma)} + \gamma(\mu) \frac{\tau(\mu,V\mu)^2}{\tau(\mu,\varsigma)}$$

for all $\mu,\varsigma \in R$. For $\mu_0\in R$, we set $\frac{\alpha(\mu_0)+\gamma (\mu_0)}{1-\beta(\mu_0)} = \lambda$. Suppose that:

$$sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\mu i+1,\mu i+2)\sigma(\mu i+1,\mu m)}{\sigma(\mu_i,\mu_{i+1})} < \frac{1}{\lambda}$$

where $\mu_{2n+1}= V\mu_{2n}$ and $\mu_{2n+2}= U\mu_{2n+1}$ for each $n\geq 0$. In addition, assume that, for every $\mu\in R$, we have $\lim_{n\to\infty} \sigma(\mu_n,\mu)$ and $\lim_{n\to\infty}\sigma(\mu,\mu_n)$ which exist and are finite. Then, V and U have a unique common fixed point.

Proof. Let $\mu_0\in R$.We construct $\{\mu_n\}$ in R by $\mu_{2n+1}=V\mu_{2n}$ and $\mu_{2n+2}=U\mu_{2n+1}$ for each $n\geq 0$. From hypothesis and (1) we obtain:

$$\tau (\mu_{2n+1},\mu_{2n+2})=\tau(V\mu_{2n},U\mu_{2n+1})$$

$$\leq \alpha(\mu_{2n})\tau(\mu_{2n},\mu_{2n+1})+\beta(\mu_{2n}) \frac{\tau(\mu_{2n},V\mu_{2n})\tau(\mu_{2n+1},U\mu_{2n+1})}{1+\tau(\mu_{2n},\mu_{2n+1})}+$$

$$\gamma(\mu) \frac{\tau(\mu_{2n},V\mu_{2n})^2}{\tau(\mu_{2n},\mu_{2n+1})}$$

$$= \alpha(\mu_{2n})\tau(\mu_{2n}, \mu_{2n+1})+\beta(\mu_{2n}) \frac{\tau(\mu_{2n},\mu_{2n+1})\tau(\mu_{2n+1},\mu_{2n+2})}{1+\tau(\mu_{2n},\mu_{2n+1})}+$$
$$\gamma(\mu_{2n}) \frac{\tau(\mu_{2n},\mu_{2n+1})^2}{\tau(\mu_{2n},\mu_{2n+1})}$$

$$\leq \alpha(\mu_{2n})\tau(\mu_{2n}, \mu_{2n+1})+\beta(\mu_{2n}) \tau(\mu_{2n+1},\mu_{2n+2})$$

$$+ \gamma(\mu_{2n})\tau(\mu_{2n},\mu_{2n+1})$$

$$= \mu_{2n-1})\tau(\mu_{2n},\mu_{2n+1})+\beta(U\mu_{2n}-1)\tau(\mu_{2n+1},\mu_{2n+2}) +\gamma(U\mu_{2n1})\tau(\mu_{2n},\mu_{2n+1})$$

$$\leq \alpha(\mu_{2n-1})\tau(\mu_{2n}, \mu_{2n+1})+\beta(\mu_{2n-1}) \tau(\mu_{2n+1},\mu_{2n+2}) + \gamma(\mu_{2n-1})\tau(\mu_{2n},\mu_{2n+1})$$

$$= \alpha(V\mu_{2n-2})\tau(\mu_{2n}, \mu_{2n+1})+\beta(V\mu_{2n-2}) \tau(\mu_{2n+1},\mu_{2n+2}) + \gamma(V\mu_{2n-2})\tau(\mu_{2n},\mu_{2n+1})$$

$$\leq \alpha(\mu_{2n-2})\tau(\mu_{2n},\mu_{2n+1})+\beta(\mu_{2n-2}) \tau(\mu_{2n+1},\mu_{2n+2}) + \gamma(\mu_{2n-2})\tau(\mu_{2n},\mu_{2n+1})$$
..
$$\leq \alpha(\mu_0)\tau(\mu_{2n}, \mu_{2n+1})+\beta(\mu_0) \tau(\mu_{2n+1},\mu_{2n+2}) + \gamma(\mu_0)\tau(\mu_{2n},\mu_{2n+1})$$

which implies that:

$$\tau(\mu_{2n+1},\mu_{2n+2}) \leq \frac{\alpha(\mu_0)+\gamma(\mu_0)}{1-\beta(\mu_0)} \tau(\mu_{2n},\mu_{2n+1})$$

similarly

$\tau(\mu_{2n+2}, \mu_{2n+3}) = \tau(U\mu_{2n+1}, V\mu_{2n+1})$

$=\tau(V\mu_{2n+2}, U\mu_{2n+1})$

$\leq \alpha(\mu_{2n+2})\tau(\mu_{2n+2},\mu_{2n+1})+\beta(\mu_{2n+2})\frac{\tau(\mu_{2n+2},V\mu_{2n+2})\tau(\mu_{2n+1},V\mu_{2n+1})}{1+\tau(\mu_{2n+2},\mu_{2n+1})}+\gamma(\mu_{2n+2})\frac{\tau(\mu_{2n+2},V\mu_{2n+2})^2}{\tau(\mu_{2n+2},\mu_{2n+1})}$

$\leq \alpha(\mu_{2n+2})\tau(\mu_{2n+2},\mu_{2n+1})+\beta(\mu_{2n+2})\frac{\tau(\mu_{2n+2},\mu_{2n+3})\tau(\mu_{2n+1},\mu_{2n+2})}{1+\tau(\mu_{2n+2},\mu_{2n+1})}+\gamma(\mu_{2n+2})\frac{\tau(\mu_{2n+2},\mu_{2n+3})^2}{\tau(\mu_{2n+2},\mu_{2n+1})}$

$= \alpha(\mu_{2n+2})\tau(\mu_{2n+2},\mu_{2n+1})+\beta(\mu_{2n+2})\tau(\mu_{2n+2},\mu_{2n+3})+\gamma(\mu_{2n+2})\tau(\mu_{2n+2},\mu_{2n+3})$

$= \alpha(U\mu_{2n+1})\tau(\mu_{2n+1}, \mu_{2n+2})+\beta(U\mu_{2n+1})\tau(\mu_{2n+2}, \mu_{2n+3})+\gamma(U\mu_{2n+1})\tau(\mu_{2n+2},\mu_{2n+3})$

$\leq \alpha(\mu_{2n+1})\tau(\mu_{2n+1}, \mu_{2n+2})\beta(\mu_{2n+1})\tau(\mu_{2n+2}, \mu_{2n+3})+\gamma(\mu_{2n+1})\tau(\mu_{2n+2},\mu_{2n+3})$

$= \alpha(V\mu_{2n})\tau(\mu_{2n+1},\mu_{2n+2})+\beta(V\mu_{2n})\tau(\mu_{2n+2},\mu_{2n+3})+\gamma(V\mu_{2n})\tau(\mu_{2n+2},\mu_{2n+3})$

$\leq \alpha(\mu_{2n})\tau(\mu_{2n+1},\mu_{2n+2})+\beta(\mu_{2n})\tau(\mu_{2n+2},\mu_{2n+3})+\gamma(\mu_{2n})\tau(\mu_{2n+2},\mu_{2n+3})$

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$\leq \alpha(\mu_0)\tau(\mu_{2n+1},\mu_{2n+2})+\beta(\mu_0)\tau(\mu_{2n+2},\mu_{2n+3})+\gamma(\mu_0)\tau(\mu_{2n+2},\mu_{2n+3})$

Which implies that:

$\tau(\mu_{2n+2},\mu_{2n+3}) \leq \frac{\alpha(\mu_0)+\gamma(\mu_0)}{1-\beta(\mu_0)}\tau(\mu_{2n+2},\mu_{2n+3}) = \lambda\,\tau(\mu_{2n+1},\mu_{2n+2})$

pursuing in this direction, we obtain:

$\tau(\mu_n,\mu_{n+1}) \leq \lambda\,\tau(\mu_{n-1},\mu_n)$

$\leq \lambda^2\tau(\mu_{n-2},\mu_{n-1})$

$\leq \ldots \lambda^n\tau(\mu_0,\mu_1)$.

Thus

$$\tau(\mu_n,\mu_{n+1}) \leq \lambda^n\tau(\mu_0,\mu_1) \quad (3)$$

For all $n, m \in N$ ($n<m$), we have:

$\tau(\mu_n, \mu_m) \leq \sigma(\mu_n,\mu_{n+1})\tau(\mu_n,\mu_{n+1})+\sigma(\mu_{n+1},\mu_m)\tau(\mu_{n+1},\mu_m)$

$\leq \sigma(\mu_n,\mu_{n+1})\tau(\mu_n,\mu_{n+1})+\sigma(\mu_{n+1},\mu_m)\sigma(\mu_{n+1},\mu_{n+2})\tau(\mu_{n+1},\mu_{n+2})+\sigma(\mu_{n+1},\mu_m)\sigma(\mu_{n+2}, \mu_m)\tau(\mu_{n+2}, \mu_m)$

$\leq \sigma(\mu_n,\mu_{n+1})\tau(\mu_n,\mu_{n+1})+\sigma(\mu_{n+1},\mu_m)\sigma(\mu_{n+1},\mu_{n+2})\tau(\mu_{n+1},\mu_{n+2})+\sigma(\mu_{n+1}, \mu_m)\sigma(\mu_{n+2}, \mu_m)\sigma(\mu_{n+2},\mu_{n+3})\tau(\mu_{n+2}, \mu_{n+3})$

$+\sigma(\mu_{n+1},\mu_m)\sigma(\mu_{n+2},\mu_m)\sigma(\mu_{n+3},\mu_m)\tau(\mu_{n+3},\mu_m) \leq \ldots$

$\leq \sigma(\mu_n,\mu_{n+1})\tau(\mu_n,\mu_{n+1})+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mu_j,\mu_m)\right)$

$\sigma(\mu_i,\mu_{i+1})\tau(\mu_i,\mu_{i+1})+\prod_{i=n+1}^{m-1}(\mu_i,\mu_m)\tau(\mu_{m-1},\mu_m)$

which further implies that:

$\tau(\mu_n, \mu_m) \leq \sigma(\mu_n,\mu_{n+1})\tau(\mu_n,\mu_{n+1})+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mu j, \mu m)\right)$

$\sigma(\mu_i,\mu_{i+1})\tau(\mu_{i,i+1})+\prod_{i=n+1}^{m-1}(\mu_{m-1},\mu_m)\tau(\mu_{m-1},\mu_m)$

$\leq \sigma(\mu_n,\mu_{n+1})\lambda^n\tau(\mu_0,\mu_1)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mu_j,\mu_m)\right)$

$\sigma(\mu_i,\mu_{i+1})\ \lambda^i\ \tau(\mu_0,\mu_1)+\prod_{i=n+1}^{m-1}(\mu_i,\mu_m)\tau(\mu_{m-1},\mu_m)$

$= \sigma\ (\mu_n,\mu_{n+1})\ \lambda^n\ \tau(\mu_0,\mu_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\ \sigma(\mu_j,\mu_m)\right)$

$\sigma(\mu_i,\mu_{i+1})\ \lambda^i\ \tau(\mu_0,\mu_1)$

Thus

$\tau(\mu_n,\ \mu_m) \leq \sigma\ (\mu_n,\mu_{n+1})\ \lambda^n\ \tau(\mu_0,\mu_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\ \sigma(\mu_j,\mu_m)\right)$

$\sigma(\mu_i,\mu_{i+1})\ \lambda^i\ \tau(\mu_0,\mu_1)$ (4)

Let

$s_l = \sum_{i=0}^{l}\left(\prod_{j=0}^{i}\ \sigma(\mu_j,\mu_m)\right)\ \sigma(\mu_i,\mu_{i+1})\ \lambda^i\ \tau(\mu_0,\mu_1)$

From (4) we obtain :

$\tau(\mu_n,\ \mu_m) \leq \sigma\ (\mu_0,\mu_1)\ [\ \lambda^n\ \sigma(\mu_i,\mu_{i+1})\ \lambda^i\ \tau(\mu_0,\mu_1)+S_{m-1}\text{-}s_n]$ (5)

Now,using the fact that σ(μ,ς)≥1,and by using the ratio test, $\lim_{n\to\infty} S_n$ exist. Hence, $\{S_n\}$ is Cauchy .Eventually,taking n,m→ ∞ in (5), we obtain that :

$\lim_{n,m\to\infty}\ \tau\ (\mu_n,\mu_m) = 0$ (6)

Hence, $\{\mu_n\}$ is a Cauchy (R, τ, σ). Thus,∃ μ^*∈R such that

$\lim_{n\to\infty}\ \tau(\mu_n,\mu^*) = 0$ (7)

that is $\mu_n \to \mu^*$as $n\to\infty$.Now, by(1)and condition (iii), we obtain:

$\tau(\mu^*,V\mu^*) \leq \sigma(\mu^*,\mu_{2n+2})\tau(\mu^*,\mu_{2n+2})+\sigma(\mu_{2n+2},V\mu^*)\tau(\mu_{2n+2},V\mu^*)$

$= \sigma(\mu^*,\mu_{2n+2})\tau(\mu^*,\mu_{2n+2})+\sigma(\mu_{2n+2},V\mu^*)\tau(U\mu_{2n+1},V\mu^*)$

$= \sigma(\mu^*,\mu_{2n+2})\tau(\mu^*,\mu_{2n+2})+\sigma(\mu_{2n+2},V\mu^*)\tau(V\mu^*,U\mu_{2n+1})$

$= \sigma(\mu^*,\mu_{2n+2})\tau(\mu^*,\mu_{2n+2})+\sigma(\mu_{2n+2},V\mu^*)\ \tau(V\mu^*,U\mu_{2n+1})$

$$\begin{bmatrix} \alpha(\mu^*)\tau(\mu^*,\mu_{2n+1}) + \beta(\mu^*)\frac{\tau(\mu^*,V\mu^*)\tau(\mu_{2n+1},\mu_{2n+2})}{1+(\mu^*,\mu_{2n+1})} \\ +\gamma(\mu^*)\frac{\tau(\mu^*,V\mu^*)^2}{\tau(\mu^*,\mu_{2n+1})} \end{bmatrix}$$

Letting $n\to\infty$ and using (7), we obtain a contradiction to $\tau\ (\mu^*,V\mu^*)>0$.Thus, $\tau(\mu^*,V\mu^*)= 0$. implies that $\mu^*=V\mu^*$. This It follows similarly that $\mu^*=U\mu^*$.Therefore, $\mu^*$is a common fixed point of V and U. Eventually, we show that $\mu^*$is a unique common fixed point of V and U. Assume that there exists another common fixed point $\mu^/$that is $\mu^/=V\mu^/=U\mu^/$. It follows from:

$$\tau(\mu^*,\mu^/) = \tau(V\mu^*,U\mu^/) \leq \alpha(\mu^*)\tau(\mu^*,\mu^/) + \beta(\mu^*)\frac{\tau(\mu^*,V\mu^*)\tau(\varsigma,U\mu^/)}{1+(\mu^*,\mu^/)} + \gamma(\mu^*)\frac{\tau(\mu^*,V\mu^*)^2}{\tau(\mu^*,\mu^/)}$$

$= \alpha\ ((\mu^*)\ \tau(\mu^*,\mu^/)$.

Since $\alpha\ (\mu^*) \in [0, 1)$, so we have $\tau(\mu^*,\mu^/)$. Therefore, we have $\mu^* = \mu^/$and thus $\mu^*$is a unique common fixed point of V and U.

Application

Theorem 5. Letting R=C ([0,1]), Now, we define T:R×R→[0,∞)

$T(\mu,\zeta) = \min_{t\in[0,1]}(\mu(t) - \zeta(t))^2$

Then, (R,σ,τ) is a complete controlled metric space with σ(μ,ς)=σ(ς,ω)=2 consider the integral equations:

$$\mu(t) = \int_0^1 K_1(t, s, \mu(s))ds + g(t) \quad (8)$$

$$\mu(t) = \int_0^1 K_2(t, s, \mu(s))ds + g(t) \quad (9)$$

for $t \in [0,1]$, where $\mu, g, h \in R$

suppose that K1, K2:[0,1]×[0,1] →R are such that $\vartheta\mu(t)$, $\theta\mu(t) \in R$, for each $\mu \in R$, where:

$$\vartheta_\mu(t) = \int_0^1 K_1(t, s, \mu(s))ds$$

and

$$\vartheta_\mu(t) = \int_0^1 K_2(t, s, \mu(s))ds$$

For all $t \in [0,1]$. If $\exists\, a,b,c$:R→[0,1) such that these assertions hold:

- $a(\vartheta_\mu+g(t)) \le a(\mu)$, $b(\vartheta_\mu+g(t)) \le b(\mu)$ and $c(\vartheta_\mu+g(t)) \le c(\mu)$
- $a(\theta_\mu+g(t)) \le a(\mu)$, $b(\theta_\mu+g(t)) \le b(\mu)$ and $c(\theta_\mu+g(t)) \le c(\mu)$
- $(a+b+c)(\mu) < 1$
- $\| \vartheta_\mu(t) - \theta_\zeta(t) + g(t) - h(t) \|^2 \le a(\mu)N_1(\mu, \zeta)(t) + b(\mu) N_2(\mu, \zeta)(t) + c(\mu) N_3(\mu, \zeta)(t)$

$\forall \mu, \varsigma \in R$, where:

$N_1(\mu, \zeta)(t) = \| \mu(t) - \zeta(t) \|^2$

$N_2(\mu, \zeta)(t) = \| \vartheta_\mu(t) - \theta(t) - \mu(t)\|^2 \, \| \theta_\zeta(t) + h(t) - \zeta(t)\|^2 /$

$1 + \| \mu(t) - \zeta(t) \|^2$

$N_3(\mu, \zeta)(t) = \| \vartheta_\mu(t) - \theta(t) - \mu(t)\|^2 / \| \mu(t) - \zeta(t) \|^2$

Consequently, there is only one common solution to the system of integral equations (8) and (9).

Proof. Define V,U: R→R by

$V\mu = \vartheta_\mu + g$

And

$U\mu = \theta_\mu + h$.

$T(V\mu, U\mu) = \max_{t\in[0,1]} (\| \vartheta_\mu(t) - \theta_\zeta(t) + g(t) - h(t) \|^2)$

$T(\mu, V\mu) = \max_{t\in[0,1]} (\| \vartheta_\mu(t) + g(t) - \mu(t) \|^2)$

$T(\zeta, U_\zeta) = \max_{t\in[0,1]} (\| \theta_\zeta(t) + h(t) - \zeta(t) \|^2)$.

It is easy to demonstrate that:

- $\alpha(V\mu) \le \alpha(\mu)$, $\beta(V\mu) \le \beta(\mu)$ and $\gamma(V\mu) \le \gamma(\mu)$
- $\alpha(U\mu) \le \alpha(\mu)$, $\beta(U\mu) \le \beta(\mu)$ and $\gamma(U\mu) \le \gamma(\mu)$
- $(\alpha+\beta+\gamma)(\mu)<1$;

- $T(V\mu, U\mu) \leq \alpha(\mu)T(\mu,\varsigma)+\beta \frac{T(\mu,V\mu)T(\varsigma,U\varsigma)}{1+T(\mu,\varsigma)} + \gamma(\mu)\frac{T(\mu,V\mu)^2}{T(\mu,\varsigma)}$

∀μ,ς ∈ r. Therefore, we can conclude that V and U share a fixed point by applying Theorem 4. As a result, there is a single point μ ∈ R where μ = Vμ = Uμ. We now have:

$$\mu = V\mu = \vartheta_{\mu} + g$$

along with $\mu = U\mu = \theta_{\mu} + h$

That's:

$$\mu(t) = \int_0^1 K_1(t, s, \mu(s))ds$$

$$\mu(t) = \int_0^1 K_2(t, s, \mu(s))ds$$

As a result, we can say that there is only one common solution to the integral equations (8) and (9).

Conclusions

We summarize our conclusions as follows:

1. To generalize the main result of Mlaiki *et al.* [3], we defined a new contractive condition in a controlled metric space by employing two control functions $\alpha, \beta : R \rightarrow [0,1)$ to the right-hand side of the inequality. Moreover, we have used a certain rational expression in the contractive condition;
2. We have taken two self-mappings instead of one self-mapping in the contractive condition of our main results;
3. We also examined integral equations as a way to apply our primary findings.

Upcoming Projects

Future research in this field will concentrate on the fixed points of fuzzy and multi-valued mappings in controlled metric spaces, with applications to fractional differential inclusion problems.

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